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Continuous Selections of the Metric Projection for 1-Chebyshev Spaces

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INTRODUCTION

If G is a nonempty subset of a normed linear space E, then for each f in E we define $P(f) := \{g_0 \in G \mid \|f - g_0\| = \inf\{\|f - g\| \mid g \in G\}\}$. This is the set of best approximations to f from G. P defines a set-valued mapping of E into 2^G which is called the *metric projection* to G. A continuous mapping s of E into G is called a *continuous selection* for the metric projection (or, more briefly, merely "a continuous selection") if $s(f)$ is in $P(f)$ for each f in E.

In this paper we examine the problem of the existence of continuous selections for n dimensional subspaces of $C[a, b]$, the Banach space of real-valued continuous functions on $[a, b]$ under the uniform norm. For a class of such spaces we give a generalization of a result of Lazar–Morris–Wulbert [3] having shown that for 1 dimensional subspaces $\langle g_0 \rangle$ of $C(X)$, X a compact Hausdorff space, there exists a continuous selection if and only if the set of zeros of g_0 has not more than one boundary point and g_0 does not change sign on X. They have raised the problem of characterizing higher dimensional spaces.

Using new methods, Nürnberger–Sommer [5, 6] proved the existence of continuous selections for a class of finite dimensional subspaces of $C[a, b]$. For their proofs they used the theory of weak Chebyshev spaces [1, 2].

Starting from these considerations we give a characterization of the existence of continuous selections for those n dimensional subspaces G of $C[a, b]$ which fulfill $\dim P(f) \leqslant 1$ for each f in $C[a, b]$. We prove that there exists a continuous selection if and only if for each g in G having zero intervals the set of zeros has not more than one boundary point and if G is weak Chebyshev.

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1. Definitions and Lemmas

In the following let G be an n dimensional subspace of $C[a, b]$. We distinguish the following zeros of a function f in $C[a, b]$.

Definition 1.1. A zero x_0 of f is said to be a *simple zero* if f changes sign at x_0 (i.e. if there exist two points $x_1 < x_0 < x_2$ in each neighborhood of x_0 such that $f(x_1) \cdot f(x_2) < 0$) or if $x_0 = a$ or $x_0 = b$. A zero x_0 on (a, b) of f is said to be a *double zero* if f does not change sign at x_0.

We denote the set of zeros of f by $Z(f)$ and the number of zeros of f by $|Z(f)|$ counting multiplicities. We define $Z(P(f)) = \{x \in [a, b] \mid g(x) = 0$ for all $g \in P(f)\}$.

In weak Chebyshev spaces there exist some distinguished best approximations—the alternation elements. These functions are very important to proving the existence of continuous selections.

Definition 1.2. G is called *weak Chebyshev* if each g in G has at most $n - 1$ changes of sign, i.e. there do not exist points $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ such that $g(x_i) \cdot g(x_{i+1}) < 0$, $i = 0,..., n - 1$.

Definition 1.3. If f is in $C[a, b]$, then g in $P(f)$ is called *alternation element* of f if there exist $n + 1$ distinct points $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ such that

$$\epsilon(-1)^i(f - g)(x_i) = \|f - g\|, \qquad i = 0,..., n, \epsilon = \pm 1.$$

The points $x_0, x_1,..., x_n$ are called *alternating extreme points.*

Jones–Karlovitz [2] have given the following characterization of weak Chebyshev subspaces.

Theorem 1.4. *G is weak Chebyshev if and only if for each f in $C[a, b]$ there exists at least one alternation element in $P(f)$.*

We describe now a property defined by Rubinstein [7]—see also [11]—which holds for subspaces of $C(Q)$, Q compact. Here we define this property only for $C[a, b]$.

Definition 1.5. G is called *k-Chebyshev* if for each f in $C[a, b]$, $P(f)$ is at most a k-dimensional polyhedron.

Rubinstein has given the following characterization of such spaces.

Theorem 1.6. *For each f in $C[a, b]$, $P(f)$ is at most a k-dimensional polyhedron if and only if every $k + 1$ linearly independent functions $g_1, g_2,..., g_{k+1}$ in G have at most $n - k - 1$ common zeros on $[a, b]$.*

We need the following lemmas.

LEMMA 1.7. (LAZAR-MORRIS-WULBERT [3]). *If s is a continuous selection of $C[a, b]$ into G and f is in $C[a, b]$, $\|f\| = 1$ and 0 is in $P(f)$, then there is a g_0 in $P(f)$ such that*

(1) *for every x in $bd\ Z(P(f)) \cap f^{-1}(1)$ and every g in $P(f)$ there is a neighborhood U of x for which $g_0 \geqslant g$ on U and*

(2) *for every x in $bd\ Z(P(f)) \cap f^{-1}(-1)$ and every g in $P(f)$ there is a neighborhood U of x for which $g_0 \leqslant g$ on U.*

LEMMA 1.8. (NÜRNBERGER-SOMMER [5]). *Lef G be weak Chebyshev. Each g in G, $g \not\equiv 0$, has at most n distinct zeros on $[a, b]$ if and only if each f in $C[a, b]$ has exactly one alternation element in $P(f)$.*

LEMMA 1.9. (SOMMER [9]). *Let G be weak Chebyshev and k-Chebyshev with $k \leqslant n - 2$. Let no g in G, $g \not\equiv 0$, have a zero interval in $[a, b]$. Then G is a Chebyshev space.*

The next lemma follows from Theorem 4.7 of Stockenberg [10].

LEMMA 1.10. *Let G be weak Chebyshev and $(n - 1)$-Chebyshev. If a function g in G, $g \not\equiv 0$, has at least n distinct zeros but no zero intervals, then $g(a) = g(b) = 0$ and g has exactly $n - 2$ distinct zeros on (a, b).*

Now we are able to give a generalization of the in the introduction formulated result of Lazar–Morris–Wulbert for 1-Chebyshev subspaces of $C[a, b]$. But it was not possible for us to get a similar result for k-Chebyshev spaces with $k > 1$.

2. THE CHARACTERIZATION THEOREM

THEOREM 2.1. *Let G be an n dimensional 1-Chebyshev subspace of $C[a, b]$. Let a function g be in G, $g \not\equiv 0$, such that g has zero intervals. Then there exists a continuous selection if and only if*

(i) *$|bd\ Z(g)| \leqslant 1$ for all g in G having zero intervals*

(ii) *G is weak Chebyshev*

Proof. (A) (1). We first show the necessity of condition (i). We assume that there exists a $g \in G$, $g \not\equiv 0$, such that g has a zero interval and $bd\ Z(g)$ has at least two points. Let $x_1, x_2 \in bd\ Z(g)$. We assume in addition that $x_1, x_2 \in \overline{\{x \in [a, b] \mid g(x) > 0\}}$ (the other cases will follow analogously).

Let $\|g\| = 1$ and $I = [y_1, y_2]$ be an interval on which g vanishes identically.

We choose n distinct points $y_1 < z_0 < z_1 < \cdots < z_{n-1} < y_2$. Now we construct a $f \in C[a, b]$ as follows:

(a) $\|f\| = 1$,

(b) $f(x_1) = 1, f(x_2) = -1, f(z_i) = (-1)^i, i = 0,..., n-1$,

(c) $\max\{-1 + g(x), -1\} \leqslant f(x) \leqslant \min\{1 + g(x), 1\}$ for all $x \in [a, b]$.

Then $\|f - g\| = 1$ and $g \in P(f)$. For if there is a $\tilde{g} \in G$ such that $\|f - \tilde{g}\| < \|f - g\|$, then $(-1)^i(\tilde{g} - g)(z_i) > 0$, $i = 0,..., n-1$. Then g and $\tilde{g}$ have at least $n - 1$ common zeros on $[y_1, y_2]$. Since G is 1-Chebyshev, this is impossible.

Therefore $P(f) = \{\alpha g \mid \alpha_1 \leqslant \alpha \leqslant \alpha_2\}$ with $\alpha_1 \leqslant 0$, $\alpha_2 \geqslant 1$. Since there exists a continuous selection, by Lemma 1.7 there is an $\alpha_0 g \in P(f)$ such that for each $\alpha g \in P(f)$ there is a neighborhood U_1 of x_1 for which $\alpha_0 g \geqslant \alpha g$ on U_1 and there is a neighborhood U_2 of x_2 for which $\alpha_0 g \leqslant \alpha g$ on U_2. Because of $x_1, x_2 \in \overline{\{x \in [a, b] \mid g(x) > 0\}}$ this is not possible. Therefore we have a contradiction.

(2) Now we show the necessity of condition (ii). We assume that there exist a $g \in G$, $\|g\| = 1$, and $n + 1$ points $a \leqslant z_0 < \cdots < z_n \leqslant b$ such that

$$g(z_i)g(z_{i+1}) < 0, \qquad i = 0,..., n-1.$$

By (1) g has no zero interval.

Therefore g has n distinct zeros $a < x_0 < x_1 < \cdots < x_{n-1} < b$ at which g changes sign.

Applying Theorem 20 of Meinardus [4] we get a $f \in C[a, b]$, $\|f\| = 1$, such that $\alpha g \in P(f)$, $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$ and $x_i \in f^{-1}(1) \cup f^{-1}(-1)$ for some $i \in \{0,..., n-1\}$. Since G is 1-Chebyshev, $P(f) = \{\alpha g \mid \alpha_1 \leqslant \alpha \leqslant \alpha_2\}$ and, therefore, $x_i \in bd\, Z(P(f)) \cap (f^{-1}(1) \cup f^{-1}(-1))$ for some $i \in \{0,..., n-1\}$.

Applying Lemma 1.7 to the points x_i we get a contradiction of the hypothesis that G has a continuous selection.

(B) Sufficiency

(1) In order to prove the converse we first show that $|Z(g)| \leqslant n - 1$ for all $g \in G$ having no zero intervals. Let $n \geqslant 3$. The case $n = 2$ is trivial. Since G is 1-Chebyshev, by (i) there are at most two linearly independent functions in G having zero intervals:

$\tilde{g}$ such that $\tilde{g} \equiv 0$ on $[a, x_1]$, $\tilde{g}(x) \neq 0$ for all $x \in (x_1, b]$,

$\tilde{\tilde{g}}$ such that $\tilde{\tilde{g}} \equiv 0$ on $[x_2, b]$, $\tilde{\tilde{g}}(x) \neq 0$ for all $x \in [a, x_2)$,

where $x_1 \leqslant x_2$.

We assume without loss of generality that $\tilde{g}$ exists. By Theorem 1.3 in [9] the space $G_1 = G|_{[a,x_1]}$ is weak Chebyshev of dimension $m \leqslant n - 1$. Since G is 1-Chebyshev, G_1 has dimension $n - 1$. If G_1 is not Chebyshev of dimension $n - 1$, then by Karlin–Studden [2] there is a $g \in G_1$, $g \not\equiv 0$ on $[a, x_1]$, such that g has $n - 1$ distinct zeros. Then the functions g and $\tilde{g}$ have $n - 1$ common zeros. But this is not possible, because G is 1-Chebyshev. If the function g exists, then the space $G_2 = G_{[x_2,b]}$ is also Chebyshev of dimension $n - 1$. If $\tilde{\tilde{g}}$ does not exist, then by Lemma 1.9 the space $G_3 = G|_{[x_1,b]}$ is Chebyshev of dimension n.

Now we assume that there is a $g \in G$ having no zero intervals such that $|Z(g)| \geqslant n$.

If g has n distinct zeros, then by Lemma 1.10 $g(a) = g(b) = 0$ and g has exactly $n - 2$ distinct zeros on (a, b). Then for some constant c there is a function $g - c\tilde{g}$ having no zero intervals such that $g - c\tilde{g}$ has n distinct zeros on $[a, b)$. Applying Lemma 1.10 we get a contradiction. Therefore we assume that g has at most $n - 1$ distinct zeros $a \leqslant y_1 < \cdots < y_r \leqslant b$, but $|Z(g)| \geqslant n$.

First case. We assume that $\tilde{\tilde{g}}$ does not exist. Since $|Z(g)| \leqslant n - 1$ on $[x_1, b]$, it is $y_1 \leqslant x_1$. If $y_1 = a$, we add to the points $y_1, \ldots, y_r$ the points $y_i + \epsilon$ for each double zero y_i and also the point $y_1 + \epsilon < x_1$. If $y_1 > a$, we add to the points $y_1, \ldots, y_r$ the points $y_i + \epsilon$ for each double zero y_i and also the point $y_1 - \epsilon$. For ϵ sufficiently small the additional points are different from $y_1, \ldots, y_r$ and contained in $[a, b]$. Furthermore, the resulting set contains at least $n + 1$ points. We arrange these in increasing order and denote the first $n + 1$ of these points by $s_0, s_1, \ldots, s_n$. The values $g(s_i)$ must then alternate in sign in the sense that $g(s_i)$ must then alternate in sign in the sense that $g(s_i) \geqslant 0$ for i odd and $g(s_i) \leqslant 0$ for i even or vice-versa. Since G_1 is Chebyshev of dimension $n - 1$, it is $s_{n-1} > x_1$.

Let $g_1, g_2, \ldots, g_n$ be a basis of G such that $\det(g_i(t_j))_{i,j=1,\ldots,n} \geqslant 0$ for all $a \leqslant t_1 < t_2 < \cdots < t_n \leqslant b$. (see Karlin–Studden [2, p. 3]).

We show: $\det(g_i(\tilde{s}_j))_{i,j=1,\cdots,n} > 0$ for all sets

$$\{\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\} \subset \{s_0, s_1, \ldots, s_n\}, \qquad \tilde{s}_1 < \tilde{s}_2 < \cdots < \tilde{s}_n .$$

If there are n points $\tilde{s}_1 < \tilde{s}_2 < \cdots < \tilde{s}_n$ such that $\det(g_i(\tilde{s}_j))_{i,j=1,\ldots,n} = 0$, then there is a function $\tilde{g} \in G$, $\tilde{g} \not\equiv 0$, having at least n distinct zeros $\tilde{s}_1, \ldots, \tilde{s}_n$. Since $\tilde{s}_1 \leqslant x_1$ and $\tilde{s}_n > x_1$, the function $\tilde{g}$ has no zero interval in $[a, b]$. But this is not possible as before shown. Hence $\det(g_i(\tilde{s}_j))_{i,j=1,\ldots,n} > 0$. Now applying the proof of Theorem 4.2 of Karlin–Studden [2] we can show that g has at least $n + 1$ distinct zeros. But this is a contradiction of the hypothesis that g has at most $n - 1$ distinct zeros.

Second case. We assume that $\tilde{\tilde{g}}$ exists. Applying the first case to the interval $[x_1, b]$ we conclude that $|Z(g)| \leqslant n - 1$ on $[x_1, b]$ and, in case $x_1 = x_2$, even $|Z(g)| \leqslant n - 2$ for all $g \in G$ having no zero intervals. Then we can conclude as in the first case and get $|Z(g)| \leqslant n - 1$ on $[a, b]$.

(2) Now we show that for each f in $C[a, b]$ all best approximations to f coincide on $[a, x_1]$ or $[x_2, b]$.

Let $f \in C[a, b]$ and $g_0 \in P(f)$ be an alternation element of f and let $a \leqslant y_0 < y_1 < \cdots < y_n \leqslant b$ be $n + 1$ alternating extreme points of $f - g_0$. Then $0 \in P(f - g_0)$ is an alternation element of $f - g_0$. If $g_1 \in P(f)$, then it follows:

$$\epsilon(-1)^i(f - g_0)(y_i) = \| f - g_0 \| \geqslant \epsilon(-1)^i(f - g_0 - (g_1 - g_0))(y_i),$$
$$i = 0, \ldots, n, \ \epsilon = \pm 1.$$

Then $\epsilon(-1)^i(g_1 - g_0)(y_i) \geqslant 0$, $i = 0, \ldots, n$, and therefore $|Z(g_1 - g_0)| \geqslant n$. By (1) the function $g_1 - g_0$ has a zero interval and by (i) there exists an interval $I = [a, x_1]$ or $I = [x_2, b]$ such that $g_1 \equiv g_0$ on I. Since G is 1-Chebyshev, all $g \in P(f)$ coincide on I.

(3) Now we choose a selection as follows: Let $f \in C[a, b]$ and $g_0 \in P(f)$. Then $0 \in P(f - g_0) = \{\alpha g \mid \alpha_1 \leqslant \alpha \leqslant \alpha_2\}$. By 2) all $g \in P(f - g_0)$ have a zero interval. Thus $g \in \langle \tilde{g} \rangle$ or $g \in \langle \tilde{\tilde{g}} \rangle$.

Let without loss of generality $g \in \langle \tilde{g} \rangle$. Since $\tilde{g}$ has no zero on $(x_1, b]$, the space $\langle \tilde{g} \rangle$ is an 1 dimensional weak Chebyshev subspace of $C[x_1, b]$. By Lemma 1.8 there exists exactly one alternation element $\bar{g}$ of $f - g_0$ for approximation by $\langle \tilde{g} \rangle$ on $[x_1, b]$.

We define: $s(f) := g_0 + g_1$, where

$$g_1 := \begin{cases} 0, & a \leqslant x \leqslant x_1, \\ \bar{g}, & x_1 \leqslant x \leqslant b. \end{cases}$$

Then it is easy to show that $g_0 + g_1 \in P(f)$ and $g_0 + g_1$ is independent of the choice of $g_0 \in P(f)$. If $g \in \langle \tilde{\tilde{g}} \rangle$, then we define $s(f)$ analogously.

(4) Now we show the continuity of this selection: We assume that there exists a sequence $(f_n) \subset C[a, b]$ such that $f_n \to f$ and $s(f_n) \to \hat{g}, \hat{g} \neq s(f)$.

Let $I = [a, x_1]$ be that interval on which all best approximations of f coincide and let, for each n, I_n be that interval on which all best approximations of f_n coincide. We can assume without loss of generality that $I_n = \tilde{I}$ for all $n \in \mathbb{N}$.

First case. $\tilde{I} = [x_2, b]$. Then $f_n - s(f_n)$ has at least two alternating extreme points in $[x_1, b]$ for each n. Otherwise, for some constant $c_n \neq 0$,

$\| f_n - s(f_n) - c_n\tilde{g} \| = \|f_n - s(f_n)\|$ and therefore $c_n\tilde{g} \in P(f_n - s(f_n))$. Because of $P(f_n - s(f_n)) = \{\alpha\tilde{g} \mid \alpha_{1n} \leqslant \alpha \leqslant \alpha_{2n}\}$ we get a contradiction. Thus $f - \hat{g}$ has also at least two alternating extreme points in $[x_1, b]$. Since $\hat{g} - g_0 \in \langle\tilde{g}\rangle$, by Theorem 1.4 the function $\hat{g} - g_0$ is an alternation element of $f - g_0$ for approximation by $\langle\tilde{g}\rangle$ on $[x_1, b]$. By Lemma 1.8 there is only one alternation element and therefore $\hat{g} - g_0 = g_1$. Thus $\hat{g} = g_0 + g_1 = s(f)$. This is a contradiction to the assumption that $\hat{g} \neq s(f)$.

Second case. $\tilde{I} = [a, x_1]$. We examine this case analogously.

Remark. If G is weak Chebyshev and 1-Chebyshev and there is no g in G, $g \not\equiv 0$, having a zero interval, then G has always a continuous selection:

If n is greater than 2, then it follows from Lemma 1.9 that G is Chebyshev on $[a, b]$ and the existence of a continuous selection is given.

If $n = 2$, then it follows from a result in [8] that G is Chebyshev on $[a, b)$ and on $(a, b]$. Therefore, by a result in [5] there exists a continuous selection.

EXAMPLE. Let $S_{n,1} := \langle 1, x, \ldots, x^n, (x - x_1)_+^n \rangle \subset C[a, b]$ with $a < x_1 < b$. This space is the $n + 2$ dimensional space of spline functions of degree n with one fixed knot. It is easy to show that $S_{n,1}$ is 1-Chebyshev. Moreover $S_{n,1}$ is weak Chebyshev ([2]). By Theorem 2.1 $S_{n,1}$ has a continuous selection. This result is also proved by Nürnberger–Sommer in [6]. The selection of Theorem 2.1 is the same as the selection of the result in [6].

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